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# The bosonic string and superstring models in $26+2$ and $10+2$ dimensional space-time, and the generalized Chern-Simons action

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Abstract: We have covariantized the lagrangians of the $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ models, which have $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ gauge symmetry in two dimensions, and studied their symmetric structures. The special property of the $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ models is the fact that all these models have an extra time coordinate in the target space-time. The $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ models coupled to two-dimensional gravity are string models in $26+2$ dimensional target space-time for bosonic string and in $10+2$ dimensional target space-time for superstring. Both string models have two time coordinates. In order to construct the covariant lagrangians of the $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ models the generalized Chern-Simons term plays an important role. The supersymmetric generalized Chern-Simons action is also proposed. The Green-Schwarz type of $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ superstring model has another fermionic local symmetry as well as $\kappa$-symmetry. The supersymmetry of target space-time is different from the standard one.

Keywords: Models of Quantum Gravity, 2D Gravity.

## Contents



## 1. Introduction

The motivation of this paper is to understand the reason why there is only one time coordinate in real world. If there are more than two time coordinates, what happens? To consider more than two time coordinates might be a clue to understand the origin of time. In order to understand this problem we in this paper study the symmetric structure of several $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ models each of which has an extra time coordinate.

There are several models which has more than two time coordinates. For example, F-theory [1], two-time physics [2, 3], and so on. F-theory is constructed as a field theory of (2,2)-brane in $10+2$ dimensional space-time, i.e. 10 space coordinates and 2 time coordinates. The two-time physics proposed by I. Bars et al. is constructed by using the field theory of multi particles. Several years ago, the author also proposed string models which have two time coordinates in ref. [5]. These models have $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ gauge symmetry in two dimensions. The target space-time of $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ bosonic string is $26+2$ dimensions, while that of $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ superstring is $10+2$ dimensions. The manifest covariant expression of these models was not found in those days. In this paper we will give the covariant lagrangians and will study the local gauge symmetries.

The $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ models are field theories defined in two-dimensional space-time. In section 2 we study the $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ bosonic models without and with two-dimensional gravity. In subsection 2.1 we introduce the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ bosonic model without gravity [4] and extend this model to the model which has manifestly $\mathrm{ISO}(D-1{,}1)$ Poincaré symmetry. In order to obtain the $\mathrm{ISO}(D-1{,}1)$ symmetry, i.e. in order to obtain the covariant expression, the generalized Chern-Simons term plays an important role. In subsection 2.2 the property of the generalized Chern-Simons term introduced in the previous subsection is discussed. In subsection 2.3 we couple the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ model to two-dimensional gravity, i.e. we obtain a kind of string theory which has $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ symmetry in two dimensions and ISO(26,2) Poincaré symmetry at the target space-time. Though this model is already proposed in ref. [5], the covariant expression given in this paper is new. In section 3 we study the supersymmetric version of the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ models without and with two-dimensional supergravity. We covariantize the lagrangian of the Neveu-Schwarz-Ramond type of $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ superstring model proposed in ref. [5]. This superstring model has ISO(10,2) Poincaré symmetry in the target space-time. The supersymmetric version of the generalized Chern-Simons lagrangian, which is necessary for the covariantization of $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ models, is proposed here. Section 4 is devoted to the Green-Schwarz type of $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ superstring model. We will show that this model has another fermionic local symmetry besides $\kappa$-symmetry. Section 5 is the discussions and the conclusions.

## 2. $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ bosonic theories

### 2.1 $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ bosonic model without gravity

In this subsection we study the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ model [4] which is described in two dimensions and has no gravitational field. The space-time coordinates are $x^m$ ($m = 0,\ 1$) and the signature of metric $\eta_{mn}$ is $(-,+)$. The fields of the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ model consists of an abelian gauge field $A_m(x)$ and several matter fields. The theory has both a vector U(1) local symmetry "$\mathrm{U}(1)_\mathrm{V}$" and an axial vector U(1) local symmetry "$\mathrm{U}(1)_\mathrm{A}$" in two dimensions, i.e. the theory is invariant under the local gauge transformation, $\delta A_m(x) = \partial_m v(x) + \varepsilon_m{}^n \partial_n v'(x)$.

In refs. [4] the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ model is proposed. The lagrangian is

$$\mathcal{L} = -i\sum_a \psi_a \sigma^m \partial_m \psi_a - \sum_a e_a A_m\, j^{am} + \pi \sum_{a,b} g_{ab}\, j^{am} j^b_m\,, \tag{2.1}$$

and the current is

$$j^{am} = \psi_a \sigma^m \psi_a\,, \tag{2.2}$$

where the indices $a$ and $b$ run 1, 2, …, $D$ ($D \geq 2$). $\psi_a$ are spinor fermionic matter fields. The charges $e_a$ and the coupling constants $g_{ab}$ satisfy the following two conditions,

$$\textbf{a)}\quad \sum_{a,b} G^{ab}\, e_a e_b = 0\,, \tag{2.3}$$

**b)** one of the eigenvalues of $G_{ab}$ is negative, others are positive, (2.4)

where

$$G_{ab} = \delta_{ab} + 2g_{ab}\,, \qquad G^{ab} = (G_{ab})^{-1}\,. \tag{2.5}$$

Both coupling constants $g_{ab}$ and $G_{ab}$ are real symmetric matrices. The lagrangian (2.1) is invariant under the following *local* transformations,

$$\delta A_m = \partial_m v + \varepsilon_m{}^n \partial_n v' \,, \tag{2.6}$$

$$\delta \psi_a = i e_a (v - \bar{\sigma} v') \psi_a \,. \tag{2.7}$$

The parameters $v(x)$ and $v'(x)$ describe $\mathrm{U}(1)_{\mathrm{V}}$ and $\mathrm{U}(1)_{\mathrm{A}}$ gauge transformations, respectively. In the model defined by the lagrangian (2.1), $\mathrm{U}(1)_{\mathrm{V}}$ gauge symmetry is manifest, on the other hand, $\mathrm{U}(1)_{\mathrm{A}}$ gauge symmetry is non-trivial because the axial-vector anomaly is cancelled non-perturbatively. The non-perturbative cancellation of the axial-vector anomaly requires the condition (2.3). This condition leads us to the fact that the matrix $G_{ab}$ is not positive definite nor negative definite. On the other hand, the unitarity, i.e. the absence of negative norm states, requires the condition that the number of negative eigenvalues of $G_{ab}$ is equal to zero or one. Together with condition (2.3), the condition (2.4) is obtained.

The bosonization leads us to the lagrangian

$$\mathcal{L} = -\frac{1}{2} \sum_{a,b} G_{ab} \partial^m \xi^a \partial_m \xi^b - \sum_a e_a A_m \, j^{am} \,, \tag{2.8}$$

and the current

$$j^{am} = \frac{1}{\sqrt{\pi}} \varepsilon^{mn} \partial_n \xi^a \,, \tag{2.9}$$

where $\xi^a$ are scalar bosonic matter fields. The lagrangian (2.8) is invariant under the local transformations (2.6) and

$$\delta \xi^a = \frac{1}{\sqrt{\pi}} \sum_b G^{ab} e_b \, v' \,. \tag{2.10}$$

Both $\mathrm{U}(1)_{\mathrm{V}}$ and $\mathrm{U}(1)_{\mathrm{A}}$ symmetries are manifest in the bosonized expression.

In the lagrangian (2.8), $G_{ab}$ can be considered to be a background metric. So, it is decomposed by using the $D$-dimensional vielbein $\zeta_a^I$ which satisfies

$$G_{ab} = \sum_{I,J} \eta_{IJ} \, \zeta_a^I \zeta_b^J \,, \qquad \sum_{a,b} G^{ab} \zeta_a^I \zeta_b^J = \eta^{IJ} \,, \tag{2.11}$$

where $\eta_{IJ}$ is a symmetric matrix defined by

$$\eta_{IJ} = \eta^{IJ} = \begin{cases} 1 & (I = J = i, \;\; i = 1, 2, \ldots, D-2) \\ -1 & (I = J = \hat{0}) \\ 1 & (I = J = \hat{1}) \\ 0 & (\text{otherwise}) \end{cases} \,. \tag{2.12}$$

The index $i$ runs $1, 2, \ldots, D-2$, while the index $I$ runs $1, 2, \ldots, D-2$, and $\hat{0}$, $\hat{1}$. In our notation, the indices $I$ and $J$ will always run over the above all values. The negative metric $\eta_{\hat{0}\hat{0}}$ comes from the property (2.4). The light-cone-like indices $\hat{+}$ and $\hat{-}$ are introduced by

$$X^{\hat{\pm}} = \frac{1}{\sqrt{2}} (X^{\hat{0}} \pm X^{\hat{1}}) \,. \tag{2.13}$$



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Then, one has, for example, $\eta_{\hat{+}\hat{-}} = \eta_{\hat{-}\hat{+}} = \eta^{\hat{+}\hat{-}} = \eta^{\hat{-}\hat{+}} = -1$ and $\eta_{\hat{+}\hat{+}} = \eta_{\hat{-}\hat{-}} = \eta^{\hat{+}\hat{+}} = \eta^{\hat{-}\hat{-}} = 0$. Without loss of generality, one can choose $\zeta_a^{\hat{-}}$ as

$$\zeta_a^{\hat{-}} = e_a\,, \tag{2.14}$$

owing to the condition (2.3). Then, the lagrangian (2.8) is rewritten as

$$\mathcal{L} = -\frac{1}{2}\,\partial^m \xi^I \partial_m \xi_I + \frac{1}{\sqrt{\pi}} \tilde{A}^m \partial_m \xi^{\hat{-}}\,, \tag{2.15}$$

where

$$\xi^I = \sum_a \zeta_a^I\, \xi^a$$

and $\tilde{A}^m = \varepsilon^{mn} A_n$. The indices $I$ and $J$ are lowered and raised by the background metric $\eta_{IJ}$ and $\eta^{IJ}$, respectively.

Using the light-cone-like indices, the local transformation (2.10) becomes

$$\delta\xi^{\hat{+}} = -\frac{1}{\sqrt{\pi}}\, v'\,, \qquad \delta\xi^I = 0 \qquad (I \neq \hat{+})\,. \tag{2.16}$$

This transformation makes us possible to take the light-cone-like gauge fixing condition,

$$\xi^{\hat{+}} = 0\,. \tag{2.17}$$

The path-integration of $\tilde{A}^m$ gives

$$\partial_m \xi^{\hat{-}} = 0\,. \tag{2.18}$$

The gauge field $\tilde{A}^m$ plays a role of a Lagrange multiplier.

The first term of the lagrangian (2.15) has Poincaré ISO$(D-1{,}1)$ global symmetry. So it is quite natural that one expects the existence of the lagrangian which has ISO$(D-1{,}1)$ global symmetry. Introducing the auxiliary field $\phi^I(x)$ and new gauge fields $\tilde{B}^{mI}(x) = \varepsilon^{mn} B_n^I(x)$ and $\tilde{C}(x) = \frac{1}{2}\varepsilon^{mn} C_{mn}(x)$, we have succeeded in extending the lagrangian (2.15) to the lagrangian which has ISO$(D-1{,}1)$ global symmetry,

$$\mathcal{L} = -\frac{1}{2}\partial^m \xi^I \partial_m \xi_I + \tilde{A}^m \phi_I \partial_m \xi^I + \tilde{B}^{mI} \partial_m \phi_I - \frac{1}{2}\tilde{C}\phi^I \phi_I\,. \tag{2.19}$$

The lagrangian (2.19) is invariant under the following local gauge transformations:

$$\begin{aligned} \delta\xi^I &= v'\phi^I\,, \\ \delta\tilde{A}_m &= \partial_m v' + \varepsilon_m{}^n \partial_n v\,, \\ \delta\phi^I &= 0\,, \\ \delta\tilde{B}_m^I &= \varepsilon_m{}^n \partial_n u^I + v'\partial_m \xi^I - v\varepsilon_m{}^n \partial_n \xi^I - \tilde{w}_m \phi^I\,, \\ \delta\tilde{C} &= \partial_m \tilde{w}^m + \partial_m v' \tilde{A}^m - v' \partial_m \tilde{A}^m\,, \end{aligned} \tag{2.20}$$

where $u^I(x)$ and $\tilde{w}^m(x) = \varepsilon^{mn} w_n(x)$ are new local gauge parameters. It should be noted that the gauge transformation (2.20) has on-shell reducibility, i.e. the transformation (2.20)

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has on-shell invariance under the following local gauge transformation,

$$
\begin{aligned}
\delta' v &= 0\,, \\
\delta' v' &= 0\,, \\
\delta' u^I &= w' \phi^I\,, \\
\delta' \tilde{w}_m &= \varepsilon_m{}^n \partial_n w'\,.
\end{aligned}
\tag{2.21}
$$

The lagrangian (2.19) is also invariant under the global transformations of ISO$(D-1{,}1)$ and scale symmetries as follows:

$$
\begin{aligned}
\delta \xi^I &= \omega^I{}_J \xi^J + a^I\,, \\
\delta \tilde{A}_m &= r \tilde{A}_m + \sum_{i=1}^{2g} \alpha_i h_m^{(i)}\,, \\
\delta \phi^I &= -r\phi^I + \omega^I{}_J \phi^J\,, \\
\delta \tilde{B}_m^I &= r \tilde{B}_m^I + \omega^I{}_J \tilde{B}_m^J + \sum_{i=1}^{2g} (\beta_i^I + \alpha_i \xi^I) h_m^{(i)}\,, \\
\delta \tilde{C} &= 2r\tilde{C}\,,
\end{aligned}
\tag{2.22}
$$

where $\omega_{IJ} = -\omega_{JI}$, $a^I$, $r$, $\alpha_i$ and $\beta_i^I$ are global parameters, and $h_m^{(i)}(x)$ are harmonic functions which satisfy $\partial^m h_m^{(i)} = \varepsilon^{mn}\partial_m h_n^{(i)} = 0$ ($i = 1, 2, \ldots, 2g$; $g$ = genus of $2D$ space-time). The lagrangian (2.19) is also invariant under the following two discrete transformations:

i) $\xi^I \to -\xi^I$, $\phi^I \to -\phi^I$, $\tilde{B}_m^I \to -\tilde{B}_m^I$, otherwise unchanged, (2.23)

ii) $\tilde{A}_m \to -\tilde{A}_m$, $\phi^I \to -\phi^I$, $\tilde{B}_m^I \to -\tilde{B}_m^I$, otherwise unchanged. (2.24)

Both discrete transformations are a kind of parity transformations of background space-time.

The equations of motion of the lagrangian (2.19) are

$$
\begin{aligned}
\partial^m \partial_m \xi^I - \partial_m (\tilde{A}^m \phi^I) &= 0\,, \\
\phi_I \partial_m \xi^I &= 0\,, \\
\tilde{A}^m \partial_m \xi^I - \partial_m \tilde{B}^{mI} - \tilde{C}\phi^I &= 0\,, \\
\partial_m \phi^I &= 0\,, \\
\phi^I \phi_I &= 0\,.
\end{aligned}
\tag{2.25}
$$

The solution of $\partial_m \phi^I = 0$ and $\phi^I \phi_I = 0$ in (2.25) is

$$
\phi^I = \begin{cases} 0 & (I = i) \\ -1/\sqrt{\pi} & (I = \hat{+}) \\ 0 & (I = \hat{-}) \end{cases}\,,
\tag{2.26}
$$

where we have used the global transformations (2.22). Though $\phi^I = 0$ is another solution, one needs a concept of fine tuning to realize this solution. In the case of the solution (2.26), the gauge transformation (2.20) makes us possible to choose the light-cone-like gauge fixing condition (2.17), and then, the lagrangian (2.19) becomes (2.15). Thus, the equivalence of the lagrangians (2.15) and (2.19) has been proved at the classical level.

## 2.2 Generalized Chern-Simons action

In this subsection we will show that a part of the lagrangian (2.19),

$$\mathcal{L}_{\mathrm{GCS}} = \tilde{B}^{mI}\partial_m\phi_I - \frac{1}{2}\tilde{C}\phi^I\phi_I \,, \tag{2.27}$$

which is introduced for covariantizing the lagrangian (2.15), is the generalized Chern-Simons lagrangian [6].

The generalized Chern-Simons lagrangian in two dimensions is defined by [6]

$$\mathcal{L}_{\mathrm{GCS}} = \mathrm{Htr}\left\{\phi(\mathrm{d}B + B^2) + \phi^2 C\right\}, \tag{2.28}$$

where

$$\phi = \frac{1}{2}\Sigma_{\hat{\alpha}}\phi^{\hat{\alpha}} \,, \qquad B = \frac{1}{2}\mathrm{d}x^m T_{\hat{a}} B_m^{\hat{a}} \,, \qquad C = \frac{1}{4}\mathrm{d}x^m\wedge\mathrm{d}x^n\,\Sigma'_{\hat{\alpha}'}C_{mn}^{\hat{\alpha}'} \,, \tag{2.29}$$

and $\mathrm{d} = \mathrm{d}x^m\partial_m$. Htr is a trace defined below. The lagrangian (2.28) has the gauge invariance under the local gauge transformation:

$$\begin{aligned}
\delta\phi &= [\phi, u] \,, \\
\delta B &= \mathrm{d}u + [B, u] - \{\phi, w\} \,, \\
\delta C &= \mathrm{d}w + \{B, w\} + [C, u] - [\phi, q] \,,
\end{aligned} \tag{2.30}$$

where

$$u = \frac{1}{2}T_{\hat{a}}u^{\hat{a}} \,, \qquad w = \frac{1}{2}\mathrm{d}x^m\Sigma'_{\hat{\alpha}'}w_m^{\hat{\alpha}'} \,, \qquad q = \frac{1}{4}\mathrm{d}x^m\wedge\mathrm{d}x^n\,T'_{\hat{a}'}q_{mn}^{\hat{a}'} \,. \tag{2.31}$$

Here, $T$, $T'$, $\Sigma$ and $\Sigma'$ satisfy a graded Lie algebra,

$$\begin{aligned}
\{\Sigma_{\hat{\alpha}}, \Sigma'_{\hat{\beta}'}\} &= f_{\hat{\alpha}\hat{\beta}'}{}^{\hat{c}}T_{\hat{c}} \,, \\
[\Sigma_{\hat{\alpha}}, T_{\hat{b}}] &= f_{\hat{\alpha}\hat{b}}{}^{\hat{c}}\Sigma_{\hat{c}} \,, \qquad [\Sigma'_{\hat{\alpha}'}, T_{\hat{b}}] = f_{\hat{\alpha}'\hat{b}}{}^{\hat{\gamma}'}\Sigma'_{\hat{\gamma}'} \,, \qquad [\Sigma_{\hat{\alpha}}, T'_{\hat{b}'}] = f_{\hat{\alpha}\hat{b}'}{}^{\hat{\gamma}'}\Sigma'_{\hat{\gamma}'} \,, \\
[T_{\hat{a}}, T_{\hat{b}}] &= f_{\hat{a}\hat{b}}{}^{\hat{c}}T_{\hat{c}} \,, \qquad [T_{\hat{a}}, T'_{\hat{b}'}] = f_{\hat{a}\hat{b}'}{}^{\hat{c}'}T'_{\hat{c}'} \,.
\end{aligned} \tag{2.32}$$

Note that $\Sigma_{\hat{\alpha}}$ and $\Sigma'_{\hat{\alpha}'}$ are not grassmannian. In Htr one can move any $T$, $T'$, $\Sigma$ or $\Sigma'$ cyclically, i.e. $\mathrm{Htr}\,(T\ldots) = \mathrm{Htr}\,(\ldots T)$, $\mathrm{Htr}\,(\Sigma\ldots) = \mathrm{Htr}\,(\ldots\Sigma)$, and so on.

In order to obtain (2.27) from (2.28), one chooses the representation,

$$\begin{aligned}
\Sigma_{\hat{\alpha}} &= \Gamma_I,\ \sigma_+[\Gamma_I, \Gamma_J] \,, \qquad & T_{\hat{a}} &= \sigma_+\Gamma_I \,, \\
\Sigma'_{\hat{\alpha}'} &= \sigma_+ \,, & T'_{\hat{a}'} &= \sigma_+ \,,
\end{aligned} \tag{2.33}$$

i.e.

$$\begin{aligned}
\phi &= \frac{1}{2}\Gamma_I\phi^I + \frac{1}{4}\sigma_+[\Gamma_I, \Gamma_J]\phi'^{IJ} \,, \\
B &= \frac{1}{2}\sigma_+\Gamma_I B_m^I\mathrm{d}x^m \,, \\
C &= \frac{1}{4}\sigma_+ C_{mn}\mathrm{d}x^m\wedge\mathrm{d}x^n \,,
\end{aligned} \tag{2.34}$$

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and

$$
\begin{aligned}
u &= \frac{1}{2}\sigma_+\Gamma_I u^I\,,\\
w &= \frac{1}{2}\sigma_+ w_m \mathrm{d}x^m\,,\\
q &= \frac{1}{4}\sigma_+ q_{mn}\mathrm{d}x^m\wedge\mathrm{d}x^n\,,
\end{aligned}
\tag{2.35}
$$

where $B_m^I = \varepsilon_{mn}\tilde{B}^{nI}$, $C_{mn} = -\varepsilon_{mn}\tilde{C}$, and $w_m = \varepsilon_{mn}\tilde{w}^n$. $\sigma_+$ and $\Gamma_I$ are the gamma matrices of SO(1,1) and SO$(D-1,1)$ respectively, and satisfy $\{\sigma_m,\sigma_n\} = 2\eta_{mn}$ and $\{\Gamma_I,\Gamma_J\} = 2\eta_{IJ}$ respectively. $\sigma_m$ is a $2\times 2$ matrix while $\Gamma_I$ is a $2^{D/2}\times 2^{D/2}$ matrix. The trace Htr is defined by $\mathrm{Htr}\,(\cdots) = \frac{-1}{2^{(D-2)/2}}\mathrm{Tr}(\sigma_-\cdots)$, where Tr is a usual trace which takes both $\sigma$-matrix and $\Gamma$-matrix into account, for example, $\mathrm{Tr}\,1 = 2\cdot 2^{D/2}$. The concrete expression for $\sigma$-matrices is

$$
\sigma_+ = \begin{pmatrix} 0 & 0\\ \sqrt{2} & 0\end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & -\sqrt{2}\\ 0 & 0\end{pmatrix}.
\tag{2.36}
$$

One finds, for example, $\mathrm{Htr}\,\sigma_+ = 4$, $\mathrm{Htr}\,(\sigma_+\Gamma_I\Gamma_J) = 4\eta_{IJ}$.

Since $B^2 = 0$, $[B,u] = 0$, $\{B,w\} = 0$, $[C,u] = 0$ and $[\phi,q] = 0$ in the case of representation (2.34), the lagrangian (2.28) becomes a simple form,

$$
\mathcal{L}_{\mathrm{GCS}} = \mathrm{Htr}\,(\phi\mathrm{d}B + \phi^2 C)\,,
\tag{2.37}
$$

and the gauge transformation (2.30) becomes

$$
\begin{aligned}
\delta\phi &= [\phi,u]\,,\\
\delta B &= \mathrm{d}u - \{\phi,w\}\,,\\
\delta C &= \mathrm{d}w\,.
\end{aligned}
\tag{2.38}
$$

The field $\phi'^{IJ}$ is decoupled because it does not appear in the lagrangian. Therefore, one can set $\phi'^{IJ} = 0$ without loss of generality, namely, $\phi = \frac{1}{2}\Gamma_I\phi^I$ and the gauge transformation of $\phi$ is $\delta\phi = 0$. The gauge parameter $q_{mn}$ is also decoupled.

### 2.3 $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ bosonic string

In this subsection we consider the $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ bosonic model coupled to $2D$ gravity, i.e. the $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ bosonic string model.

The $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ bosonic string model is described by the lagrangian (2.19) under the curved $2D$ space-time with metric $g_{mn}$, i.e.

$$
\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}g^{mn}\partial_m\xi^I\partial_n\xi_I + \tilde{A}^m\phi_I\partial_m\xi^I + \tilde{B}^{mI}\partial_m\phi_I - \frac{1}{2}\tilde{C}\phi^I\phi_I\right),
\tag{2.39}
$$

where $g(x) = \det g_{mn}(x)$. In this model the unitarity allows us two negative eigenvalues of $G_{ab}$, i.e. the condition

**b′)** two of the eigenvalues of $G_{ab}$ are negative, others are positive (2.40) is required instead of the condition (2.4) because the general coordinate transformations as well as the $\mathrm{U(1)_A}$ gauge transfomations remove one negative norm state, respectively. Therefore, the index $I$ runs 0, 1, 2, ..., $D-3$, $\widehat{0}$, $\widehat{1}$, where one has to use the background metric $\eta_{IJ}$

$$\eta_{IJ} = \eta^{IJ} = \begin{cases} -1 \ (I = J = 0) \\ \ \ 1 \ \ (I = J = i, \ \ i = 1, 2, \ldots, D-3) \\ -1 \ (I = J = \widehat{0}) \\ \ \ 1 \ \ (I = J = \widehat{1}) \\ \ \ 0 \ \ (\text{otherwise}) \end{cases} \tag{2.41}$$

instead of (2.12). There are two negative metric components $\eta_{00}$ and $\eta_{\widehat{0}\widehat{0}}$, while there is one negative metric component $\eta_{\widehat{0}\widehat{0}}$ in the $\mathrm{U(1)_V \times U(1)_A}$ bosonic model without gravity.

This lagrangian is invariant under the local $\mathrm{U(1)_V \times U(1)_A}$ transformations, the general coordinate transformations and the Weyl scaling as follows:

$$\begin{aligned} \delta\xi^I &= v'\phi^I \,, \\ \delta\tilde{A}_m &= \partial_m v' + \mathcal{E}_m{}^n \partial_n v \,, \\ \delta\phi^I &= 0 \,, \\ \delta\tilde{B}^I_m &= \mathcal{E}_m{}^n \partial_n u^I + v'\partial_m \xi^I - v\,\mathcal{E}_m{}^n \partial_n \xi^I - \tilde{w}_m \phi^I \,, \\ \delta\tilde{C} &= \nabla_m \tilde{w}^m + \partial_m v' \tilde{A}^m - v'\nabla_m \tilde{A}^m \,, \\ \delta g_{mn} &= 0 \,, \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} \delta\xi^I &= k^n \partial_n \xi^I \,, \\ \delta\tilde{A}_m &= k^n \partial_n \tilde{A}_m + \partial_m k^n \tilde{A}_n \,, \\ \delta\phi^I &= k^n \partial_n \phi^I \,, \\ \delta\tilde{B}^I_m &= k^n \partial_n \tilde{B}^I_m + \partial_m k^n \tilde{B}^I_n \,, \\ \delta\tilde{C} &= k^n \partial_n \tilde{C} + 2s\tilde{C} \,, \\ \delta g_{mn} &= k^l \partial_l g_{mn} + \partial_m k^l g_{ln} + \partial_n k^l g_{ml} - 2s g_{mn} \,, \end{aligned} \tag{2.43}$$

where $k^n(x)$ and $s(x)$ are local parameters for the general coordinate transformation and the Weyl transformation. $\mathcal{E}^{mn}(x) = \varepsilon^{mn}/\sqrt{-g(x)}$ is the anti-symmetric tensor on $2D$ curved space-time. It should be noted that the gauge transformation (2.42) has on-shell reducibility, as the same as the gauge transformation (2.20). The transformation (2.42) has on-shell invariance under the local gauge transformation,

$$\begin{aligned} \delta' v &= 0 \,, \\ \delta' v' &= 0 \,, \\ \delta' u^I &= w'\phi^I \,, \\ \delta' \tilde{w}_m &= \mathcal{E}_m{}^n \partial_n w' \,, \end{aligned} \tag{2.44}$$

which is the two-dimensional covariant form of (2.20).



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The lagrangian (2.39) is also invariant under the global transformations (2.22) and the discrete transformations (2.23) and (2.24), where $h_m^{(i)}(x)$ are harmonic functions on curved $2D$ space-time which satisfy $\nabla^m h_m^{(i)} = \mathcal{E}^{mn}\nabla_m h_n^{(i)} = 0$ $(i = 1, 2, \ldots, 2g)$. The Poincaré symmetry is ISO$(D-2,2)$.

By introducing new fields $\hat{A}^m$, $\hat{B}^{mI}$, $\hat{C}$, $\hat{g}^{mn}$, $Z$, and a new parameter $\hat{w}^m$,

$$\begin{aligned}
\hat{A}^m &= \sqrt{-g}\,\tilde{A}^m\,, & \hat{B}^{mI} &= \sqrt{-g}\,\tilde{B}^{mI}\,, & \hat{C} &= \sqrt{-g}\,\tilde{C}\,,\\
\hat{g}^{mn} &= \sqrt{-g}\,g^{mn}\,, & &Z\,, & \hat{w}^m &= \sqrt{-g}\,\tilde{w}^m\,,
\end{aligned} \tag{2.45}$$

the lagrangian (2.39) is rewritten as

$$\mathcal{L} = -\frac{1}{2}\hat{g}^{mn}\partial_m\xi^I\partial_n\xi_I + \hat{A}^m\phi_I\partial_m\xi^I + \hat{B}^{mI}\partial_m\phi_I - \frac{1}{2}\hat{C}\phi^I\phi_I + Z\,(\det\hat{g}^{mn}+1)\,, \tag{2.46}$$

and the gauge transformations (2.42) and (2.43) are rewritten as

$$\begin{aligned}
\delta\xi^I &= k^n\partial_n\xi^I + v'\phi^I\,,\\
\delta\hat{A}^m &= \partial_n(k^n\hat{A}^m) - \partial_n k^m\hat{A}^n + \hat{g}^{mn}\partial_n v' + \varepsilon^{mn}\partial_n v\,,\\
\delta\phi^I &= k^n\partial_n\phi^I\,,\\
\delta\hat{B}^{mI} &= \partial_n(k^n\hat{B}^{mI}) - \partial_n k^m\hat{B}^{nI} + \varepsilon^{mn}\partial_n u^I + v'\hat{g}^{mn}\partial_n\xi^I - v\,\varepsilon^{mn}\partial_n\xi^I - \hat{w}^m\phi^I\,,\\
\delta\hat{C} &= \partial_n(k^n\hat{C}) + \partial_m\hat{w}^m + \partial_m v'\hat{A}^m - v'\partial_m\hat{A}^m\,,\\
\delta\hat{g}^{mn} &= \partial_l(k^l\hat{g}^{mn}) - \partial_l k^m\hat{g}^{ln} - \partial_l k^n\hat{g}^{ml}\,,\\
\delta Z &= \partial_n(k^n Z)\,,
\end{aligned} \tag{2.47}$$

where $Z(x)$ is a Lagrange multiplier field. The third and the fourth terms in (2.46) is the generalized Chern-Simons term $\mathcal{L}_{\rm GCS}$ (2.27). Both the lagrangian (2.46) and the gauge transformations (2.47) are polynomials of fields and parameters.

At the quantum level, the absence of conformal anomaly requires $D = 28$. We will give the reason in the following. The conformal charge for a spin $j$ particle is $6j^2 - 6j + 1$. The gauge field give a negative sign to the conformal charge because the FP ghost field has opposite statistics. The $\mathrm{U(1)_V}\times\mathrm{U(1)_A}$ bosonic string model consists of one graviton $(j = 2)$, one photon $(j = 1)$, and $D$ scalar particles $(j = 0)$. Therefore, the total conformal charge of $\mathrm{U(1)_V}\times\mathrm{U(1)_A}$ bosonic string model is

$$\begin{aligned}
w^{(N=0)} &= 2\times(-13) + 2\times(-1) + D\times 1\\
&= D - 28\,,
\end{aligned} \tag{2.48}$$

where 2 comes from the number of components. The fields $\phi^I$, $\tilde{B}_m^I$, and $\tilde{C}$ do not contribute to the conformal charge because these fields come from the generalized Chern-Simons action which is considered to be topological. The cancellation of conformal anomaly, i.e. the existence of the Weyl symmetry at the quantum level requires $w^{(N=0)} = 0$. Thus, we obtain $D = 28$ from (2.48). The detail calculation about the quantization will be given in the paper [7]. The discussion about the spectrum will be also given there.



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## 3. $\mathrm{U(1)_V \times U(1)_A}$ supersymmetric theories

In this section we extend the previous models to the models with $N = 1$ supersymmetry. We use the (1,1) type superspace with coordinates $z^M = (x^m, \theta^\mu)$ ($m = 0, 1$; $\mu = 1, 2$). $\theta^\mu$ are fermionic spinor coordinates.

### 3.1 $\mathrm{U(1)_V \times U(1)_A}$ supersymmetric model without gravity

In the case of supersymmetric version of the $\mathrm{U(1)_V \times U(1)_A}$ model without gravity, we introduce superfields $\Xi^I(z)$, $\tilde\Psi^\alpha(z) = (\bar\sigma\Psi(z))^\alpha$, $\Phi^I(z)$, $\tilde\Pi^{\alpha I}(z) = (\bar\sigma\Pi^I(z))^\alpha$ and $\tilde\Lambda(z) = -\frac{1}{2}\bar\sigma^{\alpha\beta}\Lambda_{\alpha\beta}(z)$, instead of $\xi^I(x)$, $\tilde A^m(x)$, $\phi^I(x)$, $\tilde B^{mI}(x)$ and $\tilde C(x)$, respectively. $\Xi^I$, $\Phi^I$ and $\tilde\Lambda$ are scalar bosonic superfields, on the other hand, $\tilde\Psi^\alpha$ and $\tilde\Pi^{\alpha I}$ are spinor fermionic superfields. In ref. [5] the supersymmetric version of (2.8) is proposed. In the previous section we have succeeded in covariantizing the lagrangian of the $\mathrm{U(1)_V \times U(1)_A}$ bosonic model. So, it is now easy to obtain the covariant lagrangian of the $\mathrm{U(1)_V \times U(1)_A}$ supersymmetric model. The covariant lagrangian for the $\mathrm{U(1)_V \times U(1)_A}$ supersymmetric model without gravity is

$$\mathcal{L} = -\frac{1}{2}\mathrm{D}^\alpha\Xi^I\mathrm{D}_\alpha\Xi_I + \tilde\Psi^\alpha\Phi_I\mathrm{D}_\alpha\Xi^I + \tilde\Pi^{\alpha I}\mathrm{D}_\alpha\Phi_I - \frac{1}{2}\tilde\Lambda\Phi^I\Phi_I\,. \tag{3.1}$$

The background metric $\eta_{IJ}$ is (2.12). This lagrangian is invariant under the following local and global transformations: the local gauge transformations are

$$\begin{aligned}
\delta\Xi^I &= V'\Phi^I\,,\\
\delta\tilde\Psi_\alpha &= \mathrm{D}_\alpha V' + (\bar\sigma\mathrm{D})_\alpha V\,,\\
\delta\Phi^I &= 0\,,\\
\delta\tilde\Pi^I_\alpha &= (\bar\sigma\mathrm{D})_\alpha U^I + V'\mathrm{D}_\alpha\Xi^I - V(\bar\sigma\mathrm{D})_\alpha\Xi^I - \tilde W_\alpha\Phi^I\,,\\
\delta\tilde\Lambda &= \mathrm{D}^\alpha\tilde W_\alpha + \mathrm{D}^\alpha V'\tilde\Psi_\alpha - V'\mathrm{D}^\alpha\tilde\Psi_\alpha\,,
\end{aligned} \tag{3.2}$$

where the parameters $V(z)$ and $V'(z)$ describe super-$\mathrm{U(1)_V}$ and super-$\mathrm{U(1)_A}$ gauge transformations, respectively. The parameters $U^I(z)$ and $\tilde W_\alpha(z) = (\bar\sigma W(z))_\alpha$ are related with the symmetry of supersymmetric generalized Chern-Simons term whose property will be discussed in next subsection. The global transformations are

$$\begin{aligned}
\delta\Xi^I &= \omega^I{}_J\Xi^J + a^I\,,\\
\delta\tilde\Psi_\alpha &= r\tilde\Psi_\alpha + \sum_{i=1}^{4g}\alpha_i H^{(i)}_\alpha\,,\\
\delta\Phi^I &= -r\Phi^I + \omega^I{}_J\Phi^J\,,\\
\delta\tilde\Pi^I_\alpha &= r\tilde\Pi^I_\alpha + \omega^I{}_J\tilde\Pi^J_\alpha + \sum_{i=1}^{4g}(\beta^I_i + \alpha_i\Xi^I)H^{(i)}_\alpha\,,\\
\delta\tilde\Lambda &= 2r\tilde\Lambda\,,
\end{aligned} \tag{3.3}$$

where $\omega_{IJ} = -\omega_{JI}$, $a^I$, $r$, $\alpha_i$ and $\beta^I_i$ are all constant parameters. Poincaré symmetry is ISO($D-1$,1) as the same as that of the bosonic model in subsection 2.1. $H^{(i)}_\alpha(z)$ are harmonic functions on $2D$ superspace which satisfy $\mathrm{D}H^{(i)} = \mathrm{D}\bar{\sigma}H^{(i)} = 0$ ($i = 1, 2, \ldots, 4g$; $g$ = genus of $2D$ space-time), i.e. $H^{(i)}_\alpha = i(\sigma^m\theta)_\alpha h^{(i)}_m$ with $\partial^m h^{(i)}_m = \varepsilon^{mn}\partial_m h^{(i)}_n = 0$ ($i = 1, 2, \ldots, 2g$) and $H^{(i)}_\alpha = i\hat{\eta}^{(i)}_\alpha$ with $\sigma^m\partial_m\hat{\eta}^{(i)} = 0$ ($i = 2g+1, 2g+2, \ldots, 4g$). It should be noted that the local gauge transformation (3.2) has on-shell invariance under the following local gauge transformation,

$$\begin{aligned}
\delta' V &= 0\,,\\
\delta' V' &= 0\,,\\
\delta' U^I &= W'\Phi^I\,,\\
\delta'\tilde{W}_\alpha &= (\bar{\sigma}\mathrm{D})_\alpha W'\,.
\end{aligned} \tag{3.4}$$

The lagrangian is also invariant under the following two discrete transformations:

i) $\Xi^I \to -\Xi^I$, $\Phi^I \to -\Phi^I$, $\tilde{\Pi}^I_\alpha \to -\tilde{\Pi}^I_\alpha$, otherwise unchanged, (3.5)

ii) $\tilde{\Psi}_\alpha \to -\tilde{\Psi}_\alpha$, $\Phi^I \to -\Phi^I$, $\tilde{\Pi}^I_\alpha \to -\tilde{\Pi}^I_\alpha$, otherwise unchanged. (3.6)

Both discrete transformations are a kind of parity transformations of background space-time as the same as those of the bosonic case (2.23) and (2.24), respectively.

In the following, we give the component expression of fields and parameters. The superfields are expressed as

$$\begin{aligned}
\Xi^I &= \xi^I + i\theta\lambda^I + \frac{i}{2}\theta\theta F^I\,,\\
\tilde{\Psi}_\alpha &= i\hat{\psi}_\alpha + i\theta_\alpha X' + i(\bar{\sigma}\theta)_\alpha X + i(\sigma^m\theta)_\alpha\tilde{A}_m + \theta\theta\left(\psi - \frac{1}{2}\sigma^m\partial_m\hat{\psi}\right)_\alpha\,,\\
\Phi^I &= \phi^I + i\theta\kappa^I + \frac{i}{2}\theta\theta G^I\,,\\
\tilde{\Pi}^I_\alpha &= i\hat{\rho}^I_\alpha + i\theta_\alpha Y'^I + i(\bar{\sigma}\theta)_\alpha Y^I + i(\sigma^m\theta)_\alpha\tilde{B}^I_m + \theta\theta\left(\rho^I - \frac{1}{2}\sigma^m\partial_m\hat{\rho}^I\right)_\alpha\,,\\
\tilde{\Lambda} &= -2i\left(H + i\theta\pi + \frac{i}{2}\theta\theta\tilde{C}\right).
\end{aligned} \tag{3.7}$$

The gauge parameters are also expressed as

$$\begin{aligned}
V &= v + i\theta\mu + \frac{i}{2}\theta\theta M\,,\\
V' &= v' + i\theta\mu' + \frac{i}{2}\theta\theta M'\,,\\
U^I &= u^I + i\theta\nu^I + \frac{i}{2}\theta\theta N^I\,,\\
\tilde{W}_\alpha &= i\hat{\tau}_\alpha + i\theta_\alpha f' + i(\bar{\sigma}\theta)_\alpha f + i(\sigma^m\theta)_\alpha\tilde{w}_m + \theta\theta\left(\tau - \frac{1}{2}\sigma^m\partial_m\hat{\tau}\right)_\alpha\,.
\end{aligned} \tag{3.8}$$

The integration of lagrangian (3.1) with respect to $\mathrm{d}^2\theta$ gives

$$\begin{aligned}
\int\mathrm{d}^2\theta\,\mathcal{L} = &-\frac{1}{2}\partial^m\xi^I\partial_m\xi_I - \frac{i}{2}\lambda^I\sigma^m\partial_m\lambda_I + \frac{1}{2}F^I F_I + (\tilde{A}^m\partial_m\xi^I + i\psi\lambda^I - X'F^I)\phi_I +\\
&+\frac{i}{2}\hat{\psi}(\sigma^m\partial_m\xi^I + F^I)\kappa_I - \frac{i}{2}\hat{\psi}(\sigma^m\partial_m\phi_I + G_I)\lambda^I + \frac{i}{2}\lambda^I(X' + \bar{\sigma}X + \sigma^m\tilde{A}_m)\kappa_I +\\
&+\tilde{B}^{mI}\partial_m\phi_I + i\rho^I\kappa_I - Y'^I G_I + i\pi\kappa^I\phi_I - HG^I\phi_I - \frac{1}{2}\tilde{C}\phi^I\phi_I + \frac{i}{2}H\kappa^I\kappa_I\,.
\end{aligned} \tag{3.9}$$

The gauge transformation for each component is

$$
\begin{aligned}
\delta\xi^I &= v'\phi^I\,,\\
\delta\lambda^I_\alpha &= v'\kappa^I_\alpha + \mu'_\alpha\phi^I\,,\\
\delta F^I &= v'G^I - i\mu'\kappa^I + M'\phi^I\,,\\
\delta\hat\psi_\alpha &= (\mu' + \bar\sigma\mu)_\alpha\,,\\
\delta X &= M\,,\\
\delta X' &= M'\,,\\
\delta\tilde A_m &= \partial_m v' + \varepsilon_m{}^n\partial_n v\,,\\
\delta\psi_\alpha &= (\sigma^m\partial_m\mu')_\alpha\,,\\
\delta\phi^I &= 0\,,\\
\delta\kappa^I_\alpha &= 0\,,\\
\delta G^I &= 0\,,\\
\delta\hat\rho^I_\alpha &= (\bar\sigma\nu^I)_\alpha + \Big((v' - \bar\sigma v)\lambda^I\Big)_\alpha - \hat\tau_\alpha\phi^I\,,\\
\delta Y^I &= N^I - vF^I - \frac{i}{2}(\mu'\bar\sigma - \mu)\lambda^I - f\phi^I - \frac{i}{2}\hat\tau\bar\sigma\kappa^I\,,\\
\delta Y'^I &= v'F^I - \frac{i}{2}(\mu' - \mu\bar\sigma)\lambda^I - f'\phi^I + \frac{i}{2}\hat\tau\kappa^I\,,\\
\delta\tilde B^I_m &= \varepsilon_m{}^n\partial_n u^I + v'\partial_m\xi^I - v\,\varepsilon_m{}^n\partial_n\xi^I - \frac{i}{2}(\mu' + \mu\bar\sigma)\sigma_m\lambda^I - \tilde w_m\phi^I - \frac{i}{2}\hat\tau\sigma_m\kappa^I\,,\\
\delta\rho^I_\alpha &= v'(\sigma^m\partial_m\lambda^I)_\alpha + \frac{1}{2}\Big(\sigma^m\partial_m(v' - \bar\sigma v)\lambda^I\Big)_\alpha - \frac{1}{2}\Big((M' - \bar\sigma M)\lambda^I\Big)_\alpha +\\
&\quad + \frac{1}{2}\Big(\sigma^m(\mu' + \bar\sigma\mu)\Big)_\alpha\partial_m\xi^I + \frac{1}{2}(\mu' - \bar\sigma\mu)_\alpha F^I -\\
&\quad - \tau_\alpha\phi^I - \frac{1}{2}\Big((f' + \bar\sigma f + \sigma^m\tilde w_m)\kappa^I\Big)_\alpha + \frac{1}{2}\Big((G^I - \partial_m\phi^I\sigma^m)\hat\tau\Big)_\alpha\,,\\
\delta H &= f' - v'X' - \frac{i}{2}\mu'\hat\psi\,,\\
\delta\pi_\alpha &= \tau_\alpha - v'\psi_\alpha - \frac{1}{2}\Big((3X' - \bar\sigma X - \sigma^m\tilde A_m)\mu'\Big)_\alpha - \frac{1}{2}\Big((M' - \partial_m v'\sigma^m)\hat\psi\Big)_\alpha\,,\\
\delta\tilde C &= \partial_m\tilde w^m + \partial_m v'\tilde A^m - v'\partial_m\tilde A^m + 2i\mu'\psi - \frac{i}{2}\partial_m(\mu'\sigma^m\hat\psi) - 2M'X'\,.
\end{aligned}
\tag{3.10}
$$

Now, let us redefine some of the fields and the parameters as follows:

$$
\begin{aligned}
\tilde B^I_m - \frac{i}{2}\hat\psi\sigma_m\lambda^I &\;\to\; \tilde B^I_m\,,\\
\rho^I_\alpha + \frac{1}{2}\Big((F^I - \partial_m\xi^I\sigma^m)\hat\psi\Big)_\alpha + &\\
+\frac{1}{2}\Big((X' - \bar\sigma X - \sigma^m\tilde A_m)\lambda^I\Big)_\alpha + \pi_\alpha\phi^I + \frac{1}{2}\kappa^I_\alpha H &\;\to\; \rho^I_\alpha\,,\\
Y'^I + \frac{i}{2}\hat\psi\lambda^I + H\phi^I &\;\to\; Y'^I\,,\\
F^I - X'\phi^I &\;\to\; F^I\,,\\
\tilde C + X'^2 &\;\to\; \tilde C\,,
\end{aligned}
\tag{3.11}
$$



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for the fields and

$$
\begin{aligned}
(\mu+\bar\sigma\mu')_\alpha \;&\to\; \mu_\alpha\,,\\
\nu^I_\alpha-\Big((v-\bar\sigma v')\lambda^I\Big)_\alpha-(\bar\sigma\hat\tau)_\alpha\phi^I \;&\to\; \nu^I_\alpha\,,\\
N^I-vF^I+\frac{i}{2}(\mu-\mu'\bar\sigma)\lambda^I-f\phi^I-\frac{i}{2}\hat\tau\bar\sigma\kappa^I \;&\to\; N^I\,,\\
\hat\tau_\alpha+\hat\psi_\alpha v' \;&\to\; \hat\tau_\alpha\,,\\
f+v'X-\frac{i}{2}\mu'\bar\sigma\hat\psi \;&\to\; f\,,\\
\tilde w_m-\frac{i}{2}\mu'\sigma_m\hat\psi \;&\to\; \tilde w_m\,,\\
\tau_\alpha-v'\psi_\alpha-\frac12\Big((3X'-\bar\sigma X-\sigma_m\tilde A_m)\mu'\Big)_\alpha-\frac12\Big((M'-\partial_m v'\sigma^m)\hat\psi\Big)_\alpha \;&\to\; \tau_\alpha\,,
\end{aligned}
\tag{3.12}
$$

for the parameters. Then, the lagrangian (3.9) becomes a simple form,

$$
\begin{aligned}
\int\mathrm{d}^2\theta\,\mathcal{L} = &-\frac12\partial^m\xi^I\partial_m\xi_I-\frac{i}{2}\lambda^I\sigma^m\partial_m\lambda_I+\frac12F^IF_I+(\tilde A^m\partial_m\xi^I+i\psi\lambda^I)\phi_I+\\
&+\tilde B^{mI}\partial_m\phi_I+i\rho^I\kappa_I-Y'^IG_I-\frac12\tilde C\phi^I\phi_I\,.
\end{aligned}
\tag{3.13}
$$

The gauge transformation (3.10) also becomes simple as

$$
\begin{aligned}
\delta\xi^I &= v'\phi^I\,,\\
\delta\lambda^I_\alpha &= v'\kappa^I_\alpha+\mu'_\alpha\phi^I\,,\\
\delta F^I &= v'G^I-i\mu'\kappa^I\,,\\
\delta\tilde A_m &= \partial_m v'+\varepsilon_m{}^n\partial_n v\,,\\
\delta\psi_\alpha &= (\sigma^m\partial_m\mu')_\alpha\,,\\
\delta\phi^I &= 0\,,\\
\delta\kappa^I_\alpha &= 0\,,\\
\delta G^I &= 0\,,\\
\delta Y'^I &= v'F^I+\frac{i}{2}\hat\tau\kappa^I\,,\\
\delta\tilde B^I_m &= \varepsilon_m{}^n\partial_n u^I+v'\partial_m\xi^I-v\,\varepsilon_m{}^n\partial_n\xi^I-i\mu'\sigma_m\lambda^I-\tilde w_m\phi^I-\frac{i}{2}\hat\tau\sigma_m\kappa^I\,,\\
\delta\rho^I_\alpha &= v'(\sigma^m\partial_m\lambda^I)_\alpha-v'\psi_\alpha\phi^I+\mu'_\alpha F^I-\frac12 f(\bar\sigma\kappa^I)_\alpha-\\
&\quad-\frac12(\tilde w_m+v'\tilde A_m)(\sigma^m\kappa^I)_\alpha+\frac12\Big((G^I-\partial_m\phi^I\sigma^m)\hat\tau\Big)_\alpha\,,\\
\delta\tilde C &= \partial_m\tilde w^m+\partial_m v'\tilde A^m-v'\partial_m\tilde A^m+2i\mu'\psi\,,
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
\delta\hat{\psi}_\alpha &= (\bar{\sigma}\mu)_\alpha\,,\\
\delta X &= M\,,\\
\delta X' &= M'\,,\\
\delta\hat{\rho}^I_\alpha &= (\bar{\sigma}\nu^I)_\alpha\,,\\
\delta Y^I &= N^I\,,\\
\delta H &= f'\,,\\
\delta\pi_\alpha &= \tau_\alpha\,.
\end{aligned}
\tag{3.15}
$$

Note that no gauge fixing procedures are done at this stage, so the lagrangian (3.13) still has supersymmetry. The fields $\hat{\psi}_\alpha$, $X$, $X'$, $\hat{\rho}^I_\alpha$, $Y^I$, $H$, $\pi_\alpha$ do not exist in the lagrangian (3.13) and can be gauged away easily by using (3.15). The field redefinition (3.11) has decoupled these fields.

### 3.2 Supersymmetric generalized Chern-Simons action

In this subsection we will show that a part of the lagrangian (3.1),

$$
\mathcal{L}_{\rm sGCS} = \tilde{\Pi}^{\alpha I}\mathrm{D}_\alpha\Phi_I - \frac{1}{2}\tilde{\Lambda}\Phi^I\Phi_I\,,
\tag{3.16}
$$

is a kind of the supersymmetric version of the generalized Chern-Simons lagrangian.

We here propose the supersymmetric generalized Chern-Simons theory in two dimensions. The lagrangian of the supersymmetric generalized Chern-Simons theory is

$$
\mathcal{L}_{\rm sGCS} = \mathrm{Htr}\left\{\Phi(\mathrm{D}\bar{\sigma}\Pi + \Pi\bar{\sigma}\Pi) + \Phi^2\Lambda\right\},
\tag{3.17}
$$

where

$$
\Phi = \frac{1}{2}\Sigma_{\hat{\alpha}}\Phi^{\hat{\alpha}}\,, \qquad \Pi_\alpha = \frac{1}{2}T_{\hat{a}}\Pi^{\hat{a}}_\alpha\,, \qquad \Lambda = \frac{1}{2}\Sigma'_{\hat{\alpha}'}\Lambda^{\hat{\alpha}'}\,.
\tag{3.18}
$$

The lagrangian (3.17) is invariant under the local gauge transformation:

$$
\begin{aligned}
\delta\Phi &= [\Phi, U]\,,\\
\delta\Pi_\alpha &= \mathrm{D}_\alpha U + [\Pi_\alpha, U] - \{\Phi, \Upsilon_\alpha\}\,,\\
\delta\Lambda &= \mathrm{D}\bar{\sigma}\Upsilon + \Pi\bar{\sigma}\Upsilon + \Upsilon\bar{\sigma}\Pi + [\Lambda, U] - [\Phi, Q]\,,
\end{aligned}
\tag{3.19}
$$

where

$$
U = \frac{1}{2}T_{\hat{a}}U^{\hat{a}}\,, \qquad \Upsilon_\alpha = \frac{1}{2}\Sigma'_{\hat{\alpha}'}W^{\hat{\alpha}'}_\alpha\,, \qquad Q = \frac{1}{2}T'_{\hat{a}'}Q^{\hat{a}'}\,.
\tag{3.20}
$$

Note that $\Sigma_{\hat{\alpha}}$ and $\Sigma'_{\hat{\alpha}'}$ are not grassmannian.

By the following identification,

$$
\begin{aligned}
\mathrm{D}_1 &= \mathrm{d}x^0\partial_0\,, & \mathrm{D}_2 &= \mathrm{d}x^1\partial_1\,,\\
\Pi_1 &= \mathrm{d}x^0 B_0\,, & \Pi_2 &= \mathrm{d}x^1 B_1\,,\\
\Lambda &= \Lambda_{12} = \mathrm{d}x^0\wedge\mathrm{d}x^1 C_{01}\,,
\end{aligned}
\tag{3.21}
$$

we find that the supersymmetric version of the generalized Chern-Simons lagrangian proposed above is equivalent to the generalized Chern-Simons lagrangian. Namely, the algebraic structure of both theories are the same. In the supersymmetric generalized Chern-Simons theory the matrix $\bar{\sigma}$ plays a role of the wedge product $\wedge$ in the generalized Chern-Simons theory. The supercovariant derivative D corresponds to the external derivative d, so the property $\mathrm{D}\bar{\sigma}\mathrm{D} = 0$ corresponds to $\mathrm{d}\wedge\mathrm{d} = 0$. Thus, one can expect that $2^{D/2}$-dimensional generalized Chern-Simons theory will be deeply related with $D$-dimensional supersymmetric generalized Chern-Simons theory.

In order to obtain (3.16) from the general expression (3.17), one chooses the representation (2.33), i.e.

$$\begin{aligned}\Phi &= \frac{1}{2}\Gamma_I\Phi^I + \frac{1}{4}\sigma_+[\Gamma_I,\Gamma_J]\Phi'^{IJ}\,,\\ \Pi_\alpha &= \frac{1}{2}\sigma_+\Gamma_I\Pi^I_\alpha\,,\\ \Lambda &= \frac{1}{2}\sigma_+\tilde{\Lambda}\,,\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}U &= \frac{1}{2}\sigma_+\Gamma_I U^I\,,\\ \Upsilon_\alpha &= \frac{1}{2}\sigma_+ W_\alpha\,,\\ Q &= \frac{1}{2}\sigma_+\tilde{Q}\,,\end{aligned}\tag{3.23}$$

where $\Pi^I_\alpha = (\bar{\sigma}\tilde{\Pi}^I)_\alpha$ and $W_\alpha = (\bar{\sigma}\tilde{W})_\alpha$. As the same as in the bosonic case, $\Phi'^{IJ}$ and $\tilde{Q}$ are decoupled.

### 3.3 The gauge fixing of $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ supersymmetric model

Using the gauge transformations (3.15), one easily finds that the gauge parameters $\mu_\alpha$, $M$, $M'$, $\nu^I_\alpha$, $N^I$, $f'$, $\tau_\alpha$ gauge away the fields $\hat{\psi}_\alpha$, $X$, $X'$, $\hat{\rho}^I_\alpha$, $Y^I$, $H$, $\pi_\alpha$, respectively. By this gauge fixing the lagrangian (3.13) is unchanged because the fields $\hat{\psi}_\alpha$, $X$, $X'$, $\hat{\rho}^I_\alpha$, $Y^I$, $H$, $\pi_\alpha$ do not exist in the lagrangian. The gauge transformation (3.14) is also unchanged by the same reason.

Finally, let us perform the integration of $F^I$, $\rho^I_\alpha$, and $Y'^I$. This procedure gives simple equations of motion, $F^I = 0$, $\kappa^I_\alpha = 0$, and $G^I = 0$, because they are decoupled. Then, we find

$$\int \mathrm{d}^2\theta\,\mathcal{L} = -\frac{1}{2}\partial^m\xi^I\partial_m\xi_I - \frac{i}{2}\lambda^I\sigma^m\partial_m\lambda_I + (\tilde{A}^m\partial_m\xi^I + i\psi\lambda^I)\,\phi_I + \tilde{B}^{mI}\partial_m\phi_I - \frac{1}{2}\tilde{C}\phi^I\phi_I\,.\tag{3.24}$$

The gauge transformation for each component is

$$
\begin{aligned}
\delta\xi^I &= v'\phi^I\,,\\
\delta\lambda^I_\alpha &= \mu'_\alpha\phi^I\,,\\
\delta\tilde{A}_m &= \partial_m v' + \varepsilon_m{}^n\partial_n v\,,\\
\delta\psi_\alpha &= (\sigma^m\partial_m\mu')_\alpha\,,\\
\delta\phi^I &= 0\,,\\
\delta\tilde{B}^I_m &= \varepsilon_m{}^n\partial_n u^I + v'\partial_m\xi^I - v\,\varepsilon_m{}^n\partial_n\xi^I - i\mu'\sigma_m\lambda^I - \tilde{w}_m\phi^I\,,\\
\delta\tilde{C} &= \partial_m\tilde{w}^m + \partial_m v'\tilde{A}^m - v'\partial_m\tilde{A}^m + 2i\mu'\psi\,.
\end{aligned}
\tag{3.25}
$$

The lagrangian (3.24) does not have off-shell supersymmetry any more, but is still off-shell invariant under the gauge transformation (3.25). The remaining gauge parameters are $v$, $v'$, $\mu'_\alpha$, $u^I$, and $\tilde{w}_m$. It should be noted that the generalized Chern-Simons term in (3.24) does not have the supersymmetric counterparts. This means that the supersymmetry does not lead us to a new topological feature concerning the generalized Chern-Simons term appeared in this model.

As the same as in the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ bosonic model, using the global Poincaré ISO$(D-1,1)$ symmetry and the internal symmetry, the solution (2.26) is achieved from the equations of motion without loss of generality. Then, the lagrangian (3.24) becomes

$$
\mathcal{L} = -\frac{1}{2}\,\partial^m\xi^I\partial_m\xi_I - \frac{i}{2}\lambda^I\sigma^m\partial_m\lambda_I + \frac{1}{\sqrt{\pi}}(\tilde{A}^m\partial_m\xi^{\hat{-}} + i\psi\lambda^{\hat{-}})\,,
\tag{3.26}
$$

which is the supersymmetric version of (2.15). In the case of the solution (2.26), the gauge transformation $\delta\xi^I$ and $\delta\lambda^I_\alpha$ in (3.25) make us possible to choose the light-cone-like gauge fixing condition,

$$
\xi^{\hat{+}} = 0\,,\qquad \lambda^{\hat{+}}_\alpha = 0\,.
\tag{3.27}
$$

The gauge fields $\tilde{A}^m$ and $\psi$ play roles of Lagrange multipliers.

### 3.4 $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ Neveu-Schwarz-Ramond type superstring

In this subsection we consider the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ supersymmetric model coupled to $2D$ supergravity, which is equivalent to the Neveu-Schwarz-Ramond type [8] of $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ superstring model.

The dynamical variables of $2D$ supergravity are a vielbein $E^A = \mathrm{d}z^M E_M{}^A(z)$ and a connection $\Omega_A{}^B = \mathrm{d}z^M\Omega_M(z)\varepsilon_A{}^B$, where $A=(a,\alpha)$ ($a=+,\,-$; $\alpha=1,\,2$) is a tangent space index, and

$$
\varepsilon_A{}^B = \begin{pmatrix} \varepsilon_a{}^b & 0 \\ 0 & \frac{1}{2}(\bar{\sigma})_\alpha{}^\beta \end{pmatrix}
\tag{3.28}
$$

is the generator of SO$(1,1)$ tangent group. The kinematical constraints on the torsion $T^A = \mathcal{D}E^A = -\frac{1}{2}E^C E^B T_{BC}{}^A(z)$ are [9]

$$
T_{\beta\gamma}{}^a(z) = 2i(\sigma^a)_{\beta\gamma}\,,\qquad T_{bc}{}^a(z) = T_{\beta\gamma}{}^\alpha(z) = 0\,.
\tag{3.29}
$$

Other torsion components are determined from the constraints (3.29).



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The lagrangian for the $\mathrm{U(1)_V \times U(1)_A}$ supersymmetric model coupled to supergravity is

$$\mathcal{L} = E\left(-\frac{1}{2}\mathcal{D}^\alpha \Xi^I \mathcal{D}_\alpha \Xi_I + \tilde{\Psi}^\alpha \Phi_I \mathcal{D}_\alpha \Xi^I + \tilde{\Pi}^{\alpha I}\mathcal{D}_\alpha \Phi_I - \frac{1}{2}\tilde{\Lambda}\Phi^I\Phi_I\right), \tag{3.30}$$

where $E(z) = \mathrm{sdet}\, E_M{}^A(z) = \det E_m{}^a(z) \det E_\alpha{}^\mu(z)$. The background metric $\eta_{IJ}$ is (2.41). This lagrangian is invariant under the local super $\mathrm{U(1)_V \times U(1)_A}$ transformations, the supergeneral coordinate transformations, the local Lorentz transformations, and the super-Weyl scaling as follows:

$$\begin{aligned}
\delta\Xi^I &= V'\Phi^I , \\
\delta\tilde{\Psi}_\alpha &= \mathcal{D}_\alpha V' + (\bar{\sigma}\mathcal{D})_\alpha V , \\
\delta\Phi^I &= 0 , \\
\delta\tilde{\Pi}^I_\alpha &= (\bar{\sigma}\mathcal{D})_\alpha U^I + V'\mathcal{D}_\alpha \Xi^I - V(\bar{\sigma}\mathcal{D})_\alpha \Xi^I - \tilde{W}_\alpha \Phi^I , \\
\delta\tilde{\Lambda} &= \mathcal{D}^\alpha \tilde{W}_\alpha + \mathcal{D}^\alpha V' \tilde{\Psi}_\alpha - V'\mathcal{D}^\alpha \tilde{\Psi}_\alpha , \\
\delta E_M{}^A &= 0 , \\
\delta\Omega_M &= 0 ,
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
\delta\Xi^I &= K^N\partial_N \Xi^I , \\
\delta\tilde{\Psi}_\alpha &= K^N\partial_N \tilde{\Psi}_\alpha - \frac{1}{2}L\,(\bar{\sigma})_\alpha{}^\beta \tilde{\Psi}_\beta + \frac{1}{2}S\tilde{\Psi}_\alpha , \\
\delta\Phi^I &= K^N\partial_N \Phi^I , \\
\delta\tilde{\Pi}^I_\alpha &= K^N\partial_N \tilde{\Pi}^I_\alpha - \frac{1}{2}L\,(\bar{\sigma})_\alpha{}^\beta \tilde{\Pi}^I_\beta + \frac{1}{2}S\tilde{\Pi}^I_\alpha , \\
\delta\tilde{\Lambda} &= K^N\partial_N \tilde{\Lambda} + S\tilde{\Lambda} , \\
\delta E_M{}^a &= K^N\partial_N E_M{}^a + \partial_M K^N E_N{}^a + E_M{}^b L\,\varepsilon_b{}^a - SE_M{}^a , \\
\delta E_M{}^\alpha &= K^N\partial_N E_M{}^\alpha + \partial_M K^N E_N{}^\alpha + \frac{1}{2}E_M{}^\beta L\,(\bar{\sigma})_\beta{}^\alpha - \frac{1}{2}SE_M{}^\alpha + \frac{i}{2}E_M{}^a(\sigma_a)^{\alpha\beta}\mathcal{D}_\beta S , \\
\delta\Omega_M &= K^N\partial_N \Omega_M + \partial_M K^N \Omega_N + \partial_M L + E_M{}^a \varepsilon_a{}^b \mathcal{D}_b S + E_M{}^\alpha (\bar{\sigma})_\alpha{}^\beta \mathcal{D}_\beta S ,
\end{aligned} \tag{3.32}$$

where $K^N(z)$, $L(z)$ and $S(z)$ are local parameters for the supergeneral coordinate transformation, the local Lorentz transformations, and the super-Weyl transformation, respectively. The gauge transformation (3.31) has on-shell reducibility as the same as the gauge transformation (3.2). The transformation (3.31) has on-shell invariance under the two-dimensional covariant form of the local gauge transformation (3.4). The two-dimensional covariant form is obtained by replacing D by $\mathcal{D}$. The lagrangian (3.30) is also invariant under the global transformations (3.3) and the discrete transformations (3.5) and (3.6), where $H^{(i)}_\alpha(z)$ are harmonic functions on $2D$ superspace which satisfy $\mathcal{D}H^{(i)} = \mathcal{D}\bar{\sigma}H^{(i)} = 0$ $(i = 1, 2, \ldots, 4g)$. The Poincaré symmetry is ISO($D-2$,2).

At the quantum level, the absence of superconformal anomaly requires $D = 12$. The $\mathrm{U(1)_V \times U(1)_A}$ superstring model consists of one graviton ($j = 2$), one gravitino ($j = 3/2$), one photon ($j = 1$), one photino ($j = 1/2$), $D$ scalar particles ($j = 0$), and $D$ spinor particles ($j = 1/2$). Therefore, the total conformal charge of $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ superstring model is

$$
\begin{aligned}
w^{(N=1)} &= 2\times(-13) + 2\times\frac{11}{2} + 2\times(-1) + 2\times\left(-\frac{1}{2}\right) + D\times\left(1+\frac{1}{2}\right) \\
&= \frac{3}{2}(D-12)\,. \qquad (3.33)
\end{aligned}
$$

The fields $\Phi^I$, $\tilde{\Pi}^I_m$, and $\tilde{\Lambda}$ do not contribute to the conformal charge because these fields come from the generalized Chern-Simons action which is considered to be topological. The cancellation of superconformal anomaly, i.e. the existence of the super-Weyl symmetry at the quantum level requires $w^{(N=1)} = 0$. Thus, we obtain $D = 12$ from (3.33).

## 4. $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ Green-Schwarz type superstring

In this section we try to study the Green-Schwarz type [8] of $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ superstring model. This model has a kind of manifest supersymmetry in the background target space-time. Thus, one needs not only the background space-time coordinate $\xi^I$, but also 12 dimensional fermionic spinor coordinates $\Theta^1$ and $\Theta^2$. Similar to the Green-Schwarz superstring model, we require $\Theta^1$ and $\Theta^2$ to be scalar fields in two-dimensional field theory and Majorana-Weyl spinors in 12 dimensions. Note that it is possible to require both Majorana condition and Weyl condition in 12 dimensions at the same time owing to the existence of two time coordinates [10]. In order to obtain the supersymmetry, the number of bosonic freedoms and that of fermionic freedoms are set to be equal. So, the gauge transformation $\delta\Theta^1$ and $\delta\Theta^2$ will lead us to the light-cone-like gauge fixing conditions,

$$
\Gamma^{\hat{+}}\Theta^1 = 0\,, \qquad \Gamma^{\hat{+}}\Theta^2 = 0\,, \qquad (4.1)
$$

as the same as (2.17) and (3.27).

Since the way how to covariantize the lagrangian is the same as that in the bosonic type and that in the Neveu-Schwarz-Ramond type, we here give the final expression. The covariant lagrangian of the Green-Schwarz type of $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ superstring model is[1]

$$
\begin{aligned}
\mathcal{L} = \sqrt{-g}\Big\{ &-\frac{1}{2}g^{mn}\Pi^I_m\Pi_{nI} - i\mathcal{E}^{mn}\Pi^I_m(\Theta^1\Gamma_{IJ}\partial_n\Theta^1 - \Theta^2\Gamma_{IJ}\partial_n\Theta^2)\phi^J + \qquad (4.2)\\
&+\mathcal{E}^{mn}\Theta^1\Gamma^K{}_I\partial_m\Theta^1\Theta^2\Gamma_{KJ}\partial_n\Theta^2\phi^I\phi^J + \tilde{A}^m\phi_I\Pi^I_m + \tilde{B}^{mI}\partial_m\phi_I - \frac{1}{2}\tilde{C}\phi^I\phi_I\Big\}\,,
\end{aligned}
$$

where

$$
\Pi^I_m = \partial_m\xi^I + i(\Theta^1\Gamma^I{}_J\partial_m\Theta^1 + \Theta^2\Gamma^I{}_J\partial_m\Theta^2)\phi^J\,. \qquad (4.3)
$$

The chirality of $\Theta^1$ and $\Theta^2$ is defined as follows:

$$
\begin{aligned}
&\bar{\Gamma}\Theta^1 = \pm\Theta^1\,, \qquad \bar{\Gamma}\Theta^2 = \mp\Theta^2 \quad \cdots\cdots \quad \text{type IIA},\\
&\bar{\Gamma}\Theta^1 = \pm\Theta^1\,, \qquad \bar{\Gamma}\Theta^2 = \pm\Theta^2 \quad \cdots\cdots \quad \text{type IIB and type I}. \qquad (4.4)
\end{aligned}
$$

The Majorana conditions are also necessary to $\Theta^1$ and $\Theta^2$.

[1] A similar lagrangian is obtained in [2], but the gauge structure is different. Not only $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ symmetry but also $\pi$-symmetry play essential roles in our model.

The local gauge symmetries of the lagrangian (4.2) are not only $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ gauge symmetry, general coordinate invariance, and Weyl symmetry, but also $\kappa$-symmetry, $\pi$-symmetry, and $\lambda$-symmetry defined in the following. The $\mathrm{U}(1)_{\mathrm{V}}\times\mathrm{U}(1)_{\mathrm{A}}$ gauge transformation is

$$\begin{aligned}
\delta\Theta^1 &= 0\,,\\
\delta\Theta^2 &= 0\,,\\
\delta\xi^I &= v'\phi^I\,,\\
\delta\tilde{A}_m &= \partial_m v' + \mathcal{E}_m{}^n\partial_n v\,,\\
\delta\phi^I &= 0\,,\\
\delta\tilde{B}^I_m &= \mathcal{E}_m{}^n\partial_n u^I + v'\partial_m\xi^I + 2iv'(P_{+m}{}^n\Theta^1\Gamma^I{}_J\partial_n\Theta^1 + P_{-m}{}^n\Theta^2\Gamma^I{}_J\partial_n\Theta^2)\phi^J -\\
&\quad - v\,\mathcal{E}_m{}^n\partial_n\xi^I - \tilde{w}_m\phi^I\,,\\
\delta\tilde{C} &= \nabla_m\tilde{w}^m + \partial_m v'\tilde{A}^m - v'\nabla_m\tilde{A}^m\,,\\
\delta g_{mn} &= 0\,,
\end{aligned}\tag{4.5}$$

where $P^{mn}_{\pm}$ are projection tensors defined in (A.2). The general coordinate transformation and the Weyl transformation are

$$\begin{aligned}
\delta\Theta^1 &= k^n\partial_n\Theta^1\,,\\
\delta\Theta^2 &= k^n\partial_n\Theta^2\,,\\
\delta\xi^I &= k^n\partial_n\xi^I\,,\\
\delta\tilde{A}_m &= k^n\partial_n\tilde{A}_m + \partial_m k^n\tilde{A}_n\,,\\
\delta\phi^I &= k^n\partial_n\phi^I\,,\\
\delta\tilde{B}^I_m &= k^n\partial_n\tilde{B}^I_m + \partial_m k^n\tilde{B}^I_n\,,\\
\delta\tilde{C} &= k^n\partial_n\tilde{C} + 2s\tilde{C}\,,\\
\delta g_{mn} &= k^l\partial_l g_{mn} + \partial_m k^l g_{ln} + \partial_n k^l g_{ml} - 2sg_{mn}\,.
\end{aligned}\tag{4.6}$$

The $\kappa$-transformation and the $\pi$-transformation are fermionic transformations and have the forms,

$$\begin{aligned}
\delta\Theta^1 &= \Gamma_I\kappa^{1m}\Pi^I_m\,,\\
\delta\Theta^2 &= \Gamma_I\kappa^{2m}\Pi^I_m\,,\\
\delta\xi^I &= i(\delta\Theta^1\Gamma^I{}_J\Theta^1 + \delta\Theta^2\Gamma^I{}_J\Theta^2)\phi^J\,,\\
\delta\tilde{A}^m &= -4i\,(P^{ml}_{+}\kappa^{1n}\Gamma_I\partial_l\Theta^1 + P^{ml}_{-}\kappa^{2n}\Gamma_I\partial_l\Theta^2)\Pi^I_n\,,\\
\delta\phi^I &= 0\,,\\
\delta\tilde{B}^{mI} &= -2i\,(\delta\Theta^1\Gamma^I{}_J\Theta^1P^{mn}_{-} + \delta\Theta^2\Gamma^I{}_J\Theta^2P^{mn}_{+})\Pi^J_n -\\
&\quad - \mathcal{E}^{mn}(\delta\Theta^1\Gamma^{KI}\Theta^1\Theta^1\Gamma_{KJ}\partial_n\Theta^1 - \delta\Theta^2\Gamma^{KI}\Theta^2\Theta^2\Gamma_{KJ}\partial_n\Theta^2 -\\
&\qquad - \delta\Theta^1\Gamma_{KJ}\Theta^1\Theta^1\Gamma^{KI}\partial_n\Theta^1 + \delta\Theta^2\Gamma_{KJ}\Theta^2\Theta^2\Gamma^{KI}\partial_n\Theta^2)\phi^J +\\
&\quad + i\,\tilde{A}^m(\delta\Theta^1\Gamma^I{}_J\Theta^1 + \delta\Theta^2\Gamma^I{}_J\Theta^2)\phi^J\,,
\end{aligned}$$



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$$
\begin{aligned}
\delta\tilde{C} &= -\frac{1}{6}\,\mathcal{E}^{mn}\Big\{\delta\Theta^1\Gamma^{IJ}(\Theta^1\partial_m\Theta^1\Gamma_{IJ}\partial_n\Theta^1 + 2\partial_m\Theta^1\partial_n\Theta^1\Gamma_{IJ}\Theta^1) - \\
&\qquad\qquad - \delta\Theta^2\Gamma^{IJ}(\Theta^2\partial_m\Theta^2\Gamma_{IJ}\partial_n\Theta^2 + 2\partial_m\Theta^2\partial_n\Theta^2\Gamma_{IJ}\Theta^2)\Big\}\,, \\
\delta g^{mn} &= -8i\,(P_+^{ml}\kappa^{1n}\Gamma_I\partial_l\Theta^1 + P_-^{ml}\kappa^{2n}\Gamma_I\partial_l\Theta^2)\phi^I\,,
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
\delta\Theta^1 &= \Gamma_I\pi^1\phi^I\,, \\
\delta\Theta^2 &= \Gamma_I\pi^2\phi^I\,, \\
\delta\xi^I &= i(\delta\Theta^1\Gamma^I{}_J\Theta^1 + \delta\Theta^2\Gamma^I{}_J\Theta^2)\phi^J\,, \\
\delta\tilde{A}^m &= 4i\,(P_+^{mn}\pi^1\Gamma_I\partial_n\Theta^1 + P_-^{mn}\pi^2\Gamma_I\partial_n\Theta^2)\phi^I\,, \\
\delta\phi^I &= 0\,, \\
\delta\tilde{B}^{mI} &= -2i\,(\delta\Theta^1\Gamma^I{}_J\Theta^1P_-^{mn} + \delta\Theta^2\Gamma^I{}_J\Theta^2P_+^{mn})\Pi_n^J - \\
&\quad - \mathcal{E}^{mn}(\delta\Theta^1\Gamma^{KI}\Theta^1\Theta^1\Gamma_{KJ}\partial_n\Theta^1 - \delta\Theta^2\Gamma^{KI}\Theta^2\Theta^2\Gamma_{KJ}\partial_n\Theta^2 - \\
&\qquad - \delta\Theta^1\Gamma_{KJ}\Theta^1\Theta^1\Gamma^{KI}\partial_n\Theta^1 + \delta\Theta^2\Gamma_{KJ}\Theta^2\Theta^2\Gamma^{KI}\partial_n\Theta^2)\phi^J + \\
&\quad + i\,\tilde{A}^m(\delta\Theta^1\Gamma^I{}_J\Theta^1 + \delta\Theta^2\Gamma^I{}_J\Theta^2)\phi^J\,, \\
\delta\tilde{C} &= -\frac{1}{6}\,\mathcal{E}^{mn}\Big\{\delta\Theta^1\Gamma^{IJ}(\Theta^1\partial_m\Theta^1\Gamma_{IJ}\partial_n\Theta^1 + 2\partial_m\Theta^1\partial_n\Theta^1\Gamma_{IJ}\Theta^1) - \\
&\qquad\qquad - \delta\Theta^2\Gamma^{IJ}(\Theta^2\partial_m\Theta^2\Gamma_{IJ}\partial_n\Theta^2 + 2\partial_m\Theta^2\partial_n\Theta^2\Gamma_{IJ}\Theta^2)\Big\} + \\
&\quad + 8i\,\Pi_m^I(P_+^{mn}\pi^1\Gamma_I\partial_n\Theta^1 + P_-^{mn}\pi^2\Gamma_I\partial_n\Theta^2)\,, \\
\delta g^{mn} &= 0\,.
\end{aligned}
\tag{4.8}
$$

The $\pi$-symmetry is a new local supersymmetry which does not exist in the usual Green-Schwarz superstring. The $\lambda$-transformation is a bosonic transformation and has the form,

$$
\begin{aligned}
\delta\Theta^1 &= \lambda^{1m}\partial_m\Theta^1\,, \\
\delta\Theta^2 &= \lambda^{2m}\partial_m\Theta^2\,, \\
\delta\xi^I &= i(\delta\Theta^1\Gamma^I{}_J\Theta^1 + \delta\Theta^2\Gamma^I{}_J\Theta^2)\phi^J\,, \\
\delta\tilde{A}^m &= 0\,, \\
\delta\phi^I &= 0\,, \\
\delta\tilde{B}^{mI} &= -2i\,(\delta\Theta^1\Gamma^I{}_J\Theta^1P_-^{mn} + \delta\Theta^2\Gamma^I{}_J\Theta^2P_+^{mn})\Pi_n^J - \\
&\quad - \mathcal{E}^{mn}(\delta\Theta^1\Gamma^{KI}\Theta^1\Theta^1\Gamma_{KJ}\partial_n\Theta^1 - \delta\Theta^2\Gamma^{KI}\Theta^2\Theta^2\Gamma_{KJ}\partial_n\Theta^2 - \\
&\qquad - \delta\Theta^1\Gamma_{KJ}\Theta^1\Theta^1\Gamma^{KI}\partial_n\Theta^1 + \delta\Theta^2\Gamma_{KJ}\Theta^2\Theta^2\Gamma^{KI}\partial_n\Theta^2)\phi^J + \\
&\quad + i\,\tilde{A}^m(\delta\Theta^1\Gamma^I{}_J\Theta^1 + \delta\Theta^2\Gamma^I{}_J\Theta^2)\phi^J\,, \\
\delta\tilde{C} &= -\frac{1}{6}\,\mathcal{E}^{mn}\Big\{\delta\Theta^1\Gamma^{IJ}(\Theta^1\partial_m\Theta^1\Gamma_{IJ}\partial_n\Theta^1 + 2\partial_m\Theta^1\partial_n\Theta^1\Gamma_{IJ}\Theta^1) - \\
&\qquad\qquad - \delta\Theta^2\Gamma^{IJ}(\Theta^2\partial_m\Theta^2\Gamma_{IJ}\partial_n\Theta^2 + 2\partial_m\Theta^2\partial_n\Theta^2\Gamma_{IJ}\Theta^2)\Big\}\,, \\
\delta g^{mn} &= 0\,.
\end{aligned}
\tag{4.9}
$$

Note that $\delta g = 0$ under the $\kappa$-, $\pi$-, and $\lambda$-transformations. The local parameters $\kappa^{1m}$, $\kappa^{2m}$, $\pi^1$, and $\pi^2$ are 12-dimensional Majorana-Weyl spinors and have the opposite chirality to

$\Theta^1$ and $\Theta^2$, respectively, i.e.

$$\begin{aligned} \bar\Gamma\kappa^{1m} &= \mp\kappa^{1m}\,, & \bar\Gamma\kappa^{2m} &= \pm\kappa^{2m}\,, \\ \bar\Gamma\pi^1 &= \mp\pi^1\,, & \bar\Gamma\pi^2 &= \pm\pi^2 \quad \cdots\cdots \quad \text{type IIA}, \\ \bar\Gamma\kappa^{1m} &= \mp\kappa^{1m}\,, & \bar\Gamma\kappa^{2m} &= \mp\kappa^{2m}\,, \\ \bar\Gamma\pi^1 &= \mp\pi^1\,, & \bar\Gamma\pi^2 &= \mp\pi^2 \quad \cdots\cdots \quad \text{type IIB and type I}\,. \end{aligned} \tag{4.10}$$

The local parameters $\lambda^{1m}$ and $\lambda^{2m}$ are 12-dimensional scalars. $\kappa^{1m}$, $\kappa^{2m}$, $\lambda^{1m}$, and $\lambda^{2m}$ are vectors in two dimensions and satisfy the conditions:

$$\begin{aligned} \kappa^{1m} &= P_+^{mn}\kappa_n^1\,, & \kappa^{2m} &= P_-^{mn}\kappa_n^2\,, \\ \lambda^{1m} &= P_-^{mn}\lambda_n^1\,, & \lambda^{2m} &= P_+^{mn}\lambda_n^2\,. \end{aligned} \tag{4.11}$$

The lagrangian (4.2) also has the global symmetries: Poincaré ISO(10,2) symmetry, internal scale symmetry, and supersymmetry. The Poincaré ISO(10,2) transformation and the internal scale transformation are

$$\begin{aligned} \delta\Theta^1 &= \frac{1}{2}r\Theta^1 + \frac{1}{2}\omega_{IJ}\Gamma^{IJ}\Theta^1\,, \\ \delta\Theta^2 &= \frac{1}{2}r\Theta^2 + \frac{1}{2}\omega_{IJ}\Gamma^{IJ}\Theta^2\,, \\ \delta\xi^I &= \omega^I{}_J\xi^J + a^I\,, \\ \delta\tilde A_m &= r\tilde A_m + \sum_{i=1}^{2g}\alpha_i h_m^{(i)}\,, \\ \delta\phi^I &= -r\phi^I + \omega^I{}_J\phi^J\,, \\ \delta\tilde B_m^I &= r\tilde B_m^I + \omega^I{}_J\tilde B_m^J + \sum_{i=1}^{2g}(\beta_i^I + \alpha_i\xi^I)h_m^{(i)}\,, \\ \delta\tilde C &= 2r\tilde C\,, \\ \delta g_{mn} &= 0\,. \end{aligned} \tag{4.12}$$

As the same as in the $\mathrm{U}(1)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$ bosonic model, using the global symmetries (4.12), the solution (2.26) is achieved from the equations of motion without loss of generality. Then, the gauge transformations $\delta\Theta^1$ and $\delta\Theta^2$ in (4.8) give the light-cone-like gauge fixing condition (4.1). The supersymmetry transformation is

$$\begin{aligned} \delta\Theta^1 &= \varepsilon^1\,, \\ \delta\Theta^2 &= \varepsilon^2\,, \\ \delta\xi^I &= -i(\varepsilon^1\Gamma^I{}_J\Theta^1 + \varepsilon^2\Gamma^I{}_J\Theta^2)\phi^J\,, \\ \delta\tilde A_m &= 0\,, \\ \delta\phi^I &= 0\,, \end{aligned}$$

$$
\begin{aligned}
\delta\tilde{B}^I_m &= 2i(\varepsilon^1\Gamma^I{}_J\Theta^1 P_{-m}{}^n + \varepsilon^2\Gamma^I{}_J\Theta^2 P_{+m}{}^n)\Pi^J_n + \\
&\quad + \frac{1}{3}\mathcal{E}_m{}^n(\,5\varepsilon^1\Gamma^{KI}\Theta^1\Theta^1\Gamma_{KJ}\partial_n\Theta^1 - 5\varepsilon^2\Gamma^{KI}\Theta^2\Theta^2\Gamma_{KJ}\partial_n\Theta^2 - \\
&\qquad\qquad - \varepsilon^1\Gamma_{KJ}\Theta^1\Theta^1\Gamma^{KI}\partial_n\Theta^1 + \varepsilon^2\Gamma_{KJ}\Theta^2\Theta^2\Gamma^{KI}\partial_n\Theta^2)\phi^J - \\
&\quad - i\tilde{A}_m(\varepsilon^1\Gamma^I{}_J\Theta^1 + \varepsilon^2\Gamma^I{}_J\Theta^2)\phi^J\,, \\
\delta\tilde{C} &= \frac{1}{18}\mathcal{E}^{mn}\Big\{\varepsilon^1\Gamma^{IJ}(\Theta^1\partial_m\Theta^1\Gamma_{IJ}\partial_n\Theta^1 + 2\partial_m\Theta^1\partial_n\Theta^1\Gamma_{IJ}\Theta^1) - \\
&\qquad\qquad - \varepsilon^2\Gamma^{IJ}(\Theta^2\partial_m\Theta^2\Gamma_{IJ}\partial_n\Theta^2 + 2\partial_m\Theta^2\partial_n\Theta^2\Gamma_{IJ}\Theta^2)\Big\}\,, \\
\delta g_{mn} &= 0\,.
\end{aligned}
\tag{4.13}
$$

Performing the supersymmetry transformation (4.13) twice, one obtains

$$
\delta\xi^I = -i(\varepsilon^1\Gamma^I{}_J\varepsilon'^1 + \varepsilon^2\Gamma^I{}_J\varepsilon'^2)\phi^J\,. \tag{4.14}
$$

This means that the background supersymmetry in this model is slightly different from the standard super-Poincaré ISO(10,2) symmetry because there is no translation to the direction $\phi^I$.

Finally, we summarize the global symmetries in the following.

i) $\xi^I \to -\xi^I$, $\phi^I \to -\phi^I$, $\tilde{B}^I_m \to -\tilde{B}^I_m$, otherwise unchanged, (4.15)

ii) $\Theta^1 \to -\Theta^1$, otherwise unchanged, (4.16)

iii) $\Theta^2 \to -\Theta^2$, otherwise unchanged. (4.17)

It should be noted that there is no counterpart of the discrete symmetry (2.24) or (3.6) in the Green-Schwarz type of $\mathrm{U(1)_V{\times}U(1)_A}$ superstring. This fact means unfortunately that it is not so straightforward to understand the equivalence of the Neveu-Schwarz-Ramond type and the Green-Schwarz type of $\mathrm{U(1)_V{\times}U(1)_A}$ superstring. Anyway, in order to show the equivalence, one needs to perform the quantization of both theories.

## 5. Discussions and conclusions

The form of lagrangian (2.46) suggests us that the lagrangian (2.46) will be naturally extended to the lagrangian of higher dimensional object like membrane or $p$-brane. For example, $\partial_m\phi^I = 0$, $\phi_I\partial_m\xi^I = 0$ and $\phi^I\phi_I = 0$ in (2.25) are obtained by the compactification on the internal space-time with condition $\phi^I = \partial_{\hat{+}}\xi^I$, where $\phi^I$ is constant. The $\pi$-symmetry is considered to be the third component of the $\kappa$-symmetry. Thus, the extension from $\mathrm{U(1)_V{\times}U(1)_A}$ string to (2,2)-brane[2] is one of the natural ways.

It is easy to extend the supersymmetry to $N = 2$ supersymmetry. In the case of the $N = 2$ $\mathrm{U(1)_V{\times}U(1)_A}$ superstring model, the total conformal charge is

$$
\begin{aligned}
w^{(N=2)} &= 2\times(-13)+4\times\frac{11}{2} + 2\times(-1) + 2\times(-1)+4\times\left(-\frac{1}{2}\right) + 2\times(-1) + 2D\times\left(1+\frac{1}{2}\right) \\
&= 3(D-4)\,.
\end{aligned}
\tag{5.1}
$$

[2]The world-sheet swept by string is the space-time with metric (−,+), on the other hand, the world-volume swept by (2,2)-brane is the space-time with metric (−,−,+,+).

The cancellation of superconformal anomaly, i.e. the super-Weyl symmetry at the quantum level requires $w^{(N=2)} = 0$. Thus, we obtain $D = 4$ from (5.1). It should be noted that the background space-time has two time coordinates, i.e. $D = 2 + 2 \neq 3 + 1$.

The followings are the conclusions. We have succeeded in obtaining the covariant expression of $\mathrm{U(1)_V}{\times}\mathrm{U(1)_A}$ models proposed in ref. [5], which are bosonic and supersymmetric models without and with (super)gravity. The $\mathrm{U(1)_V}\times\mathrm{U(1)_A}$ bosonic and supersymmetric models without gravity have ISO($D-1$,1) Poincaré symmetry ($D \geq 2$). The $\mathrm{U(1)_V}{\times}\mathrm{U(1)_A}$ bosonic and supersymmetric models with gravity, i.e. the $\mathrm{U(1)_V}{\times}\mathrm{U(1)_A}$ bosonic string and superstring models have ISO(26,2) and ISO(10,2) Poincaré symmetry, respectively. We also obtain the Green-Schwarz type of $\mathrm{U(1)_V}{\times}\mathrm{U(1)_A}$ superstring model. It is also possible to construct the $\mathrm{U(1)_V}\times\mathrm{U(1)_A}$ heterotic superstring by using usual method. In any cases the generalized Chern-Simons term plays an important role to covariantize the lagrangian.

The form of covariantized lagrangian in the $\mathrm{U(1)_V}\times\mathrm{U(1)_A}$ (super)string models suggests that these models are defined naturally by more than two-dimensional field theories, namely, that membrane or $p$-brane is more fundamental than string in these models. The $\mathrm{U(1)_V}{\times}\mathrm{U(1)_A}$ string models are the first examples which suggest higher dimensional object like membrane or $p$-brane in the framework of perturbative field theory without using the concept of "string duality".

The relation between the $\mathrm{U(1)_V}\times\mathrm{U(1)_A}$ superstring model, M-theory, and F-theory is unfortunately still unclear. Though M-theory is a kind of membrane theory, it does not have ISO(10,2) Poincaré symmetry. So, the $\mathrm{U(1)_V}\times\mathrm{U(1)_A}$ superstring model will be directly related with F-theory which has ISO(10,2) Poincaré symmetry and is considered to be based on (2,2)-brane.


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## A. Notations

The two-dimensional space-time indices $m$, $n$ run 0 and 1. The two-dimensional flat metric and the anti-symmetric tensor are

$$\eta_{mn} = \eta^{mn} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon^{mn} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.1}$$

In the curved two-dimensional space-time, the metric is $g_{mn}$ and the anti-symmetric tensor is $\mathcal{E}^{mn} = \varepsilon^{mn}/\sqrt{-g}$, where $g = \det g_{mn}$. The decomposition of vectors to self-dual and anti-self-dual pieces is achieved using the projection tensors

$$P_{\pm}^{mn} = \frac{g^{mn} \pm \mathcal{E}^{mn}}{2}\,, \tag{A.2}$$

which satisfy the projection conditions, $P_{\pm}^{m}{}_{k}P_{\pm}^{kn} = P_{\pm}^{mn}$ and $P_{\pm}^{m}{}_{k}P_{\mp}^{kn} = 0$. The relation $P_{\pm}^{kl}P_{\pm}^{mn} = P_{\pm}^{kn}P_{\pm}^{ml}$ is also useful. The covariant derivative $\nabla_m$ operates to fields as

$$\begin{aligned}\nabla_m\phi &= \partial_m\phi\,,\\ \nabla_m A_n &= \partial_m A_n - \Gamma^l{}_{mn}A_l\,,\\ \nabla_m A^n &= \partial_m A^n + \Gamma^n{}_{ml}A^l\,,\end{aligned}\tag{A.3}$$

where $\Gamma^l{}_{mn} = \frac{1}{2}g^{lk}(\partial_m g_{kn} + \partial_n g_{mk} - \partial_k g_{mn})$.

The spinor metric is

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.\tag{A.4}$$

The spinor indices are raised or lowered as

$$\theta^\alpha = \eta^{\alpha\beta}\theta_\beta\,,\qquad \theta_\alpha = \theta^\beta\eta_{\beta\alpha}\,.\tag{A.5}$$

The $\sigma$-matrices satisfy

$$\{\sigma^m,\sigma^n\} = 2\eta^{mn}\,.\tag{A.6}$$

The explicit expression of $\sigma$-matrices is

$$(\sigma^0)_\alpha{}^\beta = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\qquad (\sigma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\tag{A.7}$$

and

$$(\bar\sigma)_\alpha{}^\beta = (\sigma^0\sigma^1)_\alpha{}^\beta = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.\tag{A.8}$$

The inner-product of spinors is defined by

$$\theta\mathcal{M}_\sigma\chi = \theta^\alpha(\mathcal{M}_\sigma)_\alpha{}^\beta\chi_\beta\,,\tag{A.9}$$

where $\mathcal{M}_\sigma$ represents any product of $\sigma$-matrices. The integration of spinor coordinates is

$$\int \mathrm{d}^2\theta = \frac{1}{2}\int \mathrm{d}\theta^1\mathrm{d}\theta^2\,.\tag{A.10}$$

The flat super-covariantderivative is

$$\mathrm{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^m\theta)_\alpha\partial_m\,.\tag{A.11}$$

In the curved super-space-time, the super-covariantderivative $\mathcal{D}_\alpha = E_\alpha{}^M\mathcal{D}_M$ operates to superfields as

$$\begin{aligned}\mathcal{D}\Phi &= \mathrm{d}\Phi\,,\\ \mathcal{D}\Psi_A &= \mathrm{d}\Psi_A + \Omega\,\varepsilon_A{}^B\Psi_B\,,\\ \mathcal{D}\Psi^A &= \mathrm{d}\Psi^A - \Omega\,\Psi^B\varepsilon_B{}^A\,,\end{aligned}\tag{A.12}$$

where $\mathcal{D} = \mathrm{d}z^M\mathcal{D}_M$, $\mathrm{d} = \mathrm{d}z^M\partial_M$ and $\Omega = \mathrm{d}z^M\Omega_M$.



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The $D$-dimensional space-time indices $I$, $J$ run $1, \ldots, D-2, \widehat{0}, \widehat{1}$ in the case of $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ (supersymmetric) models without gravity, and $0, 1, \ldots, D-3, \widehat{0}, \widehat{1}$ in the case of $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ (supersymmetric) models coupled to gravity. The $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ (supersymmetric) models coupled to gravity are equivalent to the $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ (super)string models. $D$-dimensional space-time flat metric is

$$\eta_{IJ} = \eta^{IJ} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{D-2}, -1, 1) \tag{A.13}$$

in the case of $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ (supersymmetric) models without gravity, and

$$\eta_{IJ} = \eta^{IJ} = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{D-3}, -1, 1) \tag{A.14}$$

in the case of $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ (super)string models. The value of $D$ is $D \geq 2$ for $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ (supersymmetric) models without gravity, $D = 28$ for $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ bosonic string model, and $D = 12$ for $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ superstring model.

The 12-dimensional $\Gamma$-matrices are $64 \times 64$ matrices and satisfy

$$\{\Gamma^I, \Gamma^J\} = 2\eta^{IJ}\,, \qquad (\Gamma^I)^\dagger = \Gamma_I\,. \tag{A.15}$$

The explicit expression of $\Gamma$-matrices is

$$\begin{aligned} \Gamma^{\widehat{0}} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma^{\widehat{1}} = \begin{pmatrix} 0 & \bar{\gamma} \\ \bar{\gamma} & 0 \end{pmatrix}, \\ \Gamma^{0} &= \begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix}, \qquad \Gamma^{i} = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix}, \end{aligned} \tag{A.16}$$

and

$$\bar{\Gamma} = \Gamma^{\widehat{0}}\Gamma^{\widehat{1}}\Gamma^0\Gamma^1 \cdots \Gamma^9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \bar{\gamma} = \gamma^0\gamma^1 \cdots \gamma^9\,, \tag{A.17}$$

where $\gamma^0$, $\gamma^i$, $\bar{\gamma}$ are 10-dimensional $32 \times 32$ $\gamma$-matrices which satisfy $(\gamma^0)^2 = -1$, $\{\gamma^0, \gamma^i\} = 0$, $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. The inner-product of 12-dimensional spinors is defined by

$$\Theta \mathcal{M}_\Gamma \Psi = \Theta^\dagger \Gamma^{\widehat{0}} \Gamma^0 \mathcal{M}_\Gamma \Psi\,, \tag{A.18}$$

where $\mathcal{M}_\Gamma$ represents any product of $\Gamma$-matrices. If we choose the expression of $\Gamma$-matrices which satisfy $(\Gamma^I)^* = \Gamma^I$, then the Majorana condition for a spinor $\Psi$ becomes the simplest form, $\Psi^* = \Psi$. The following relations are useful to check the local gauge symmetries of the Green-Schwarz type of $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ superstring model:

$$\Gamma^K\Gamma^{IJ} = \Gamma^{KIJ} + \eta^{KI}\Gamma^J - \eta^{KJ}\Gamma^I\,, \tag{A.19}$$

$$\Theta\Gamma^{I_1 \ldots I_n}\Psi = \Psi\Gamma^{I_n \ldots I_1}\Theta\,, \tag{A.20}$$

$$\begin{aligned} &\Theta\Gamma^{KI}(\Psi_1\Psi_2\Gamma_K{}^J\Psi_3 + \Psi_2\Psi_3\Gamma_K{}^J\Psi_1 + \Psi_3\Psi_1\Gamma_K{}^J\Psi_2) + \{I \leftrightarrow J\} = \\ &\quad = \frac{1}{16}\eta^{IJ}\Big(7\Theta\Gamma^{KL}\Psi_1\Psi_2\Gamma_{KL}\Psi_3 - \frac{1}{6!}\Theta\Gamma^{KLMNPQ}\Psi_1\Psi_2\Gamma_{KLMNPQ}\Psi_3\Big)\,, \end{aligned} \tag{A.21}$$

where $\Gamma^{IJ\cdots}$ are defined by

$$\Gamma^{IJ} = \frac{1}{2}(\Gamma^I\Gamma^J - \Gamma^J\Gamma^I)\,, \tag{A.22}$$

$$\Gamma^{IJK} = \frac{1}{6!}(\Gamma^I\Gamma^J\Gamma^K - \Gamma^J\Gamma^I\Gamma^K + \{\text{cyclic permutation}\})\,, \tag{A.23}$$

and so on.

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